

# Cosmological models with $\Omega_M$ -dependent cosmological constant

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## Abstract

We investigate the evolution of the scale factor in a cosmological model in which the cosmological constant is given by the scalar arisen by the contraction of the stress-energy tensor.

## 1 Introduction

A revolutionary development seems to take place in cosmology during the last few years. The evidence continues to mount that the expansion of the universe is accelerating rather than slowing down. New observation suggests a universe that is light-weight, is accelerating, and is flat [10] [1] [6]. To induce cosmic acceleration it is necessary to consider some components, whose equations of state are different from baryons, neutrinos, dark matter, or radiation considered in the standard cosmology.

As it is well-known, one way to account for cosmic acceleration is the introduction a new type of energy, the so-called *quintessence* ("dark energy"), a dynamical, spatially inhomogeneous form of energy with negative pressure [13]. A common example is the

energy of a slowly evolving scalar field with positive potential energy, similar to the inflation field in the inflation cosmology. The quintessence cosmological scenario (QCDM) is a spatially flat FRW space-time dominated by the radiation at early times, and cold dark matter (CDM) and quintessence (Q) later time. A series of papers of Steinhardt et al. is devoted to the various quintessence cosmological models [12] (a number of follow-up studies are underway). The quintessence is supposed to obey an equation of state of the form

$$p_Q c^{-2} = w_Q \rho_Q, \quad -1 < w_Q < 0. \quad (1)$$

In many models  $w_Q$  can vary over time. For the vacuum energy (static cosmological constant), it holds  $w_Q = -1$  and  $\dot{w}_Q = 0$ .

In what follows we present a variant of the quintessence cosmological scenario in which the content of black energy is given by the cosmological constant. Like that of many other features of relativistic cosmology, history of the static and dynamic cosmological constant in Einstein's equations abounds in peculiarities and paradoxes. The question is of fundamental significance in present-day cosmology and its discussion raises fundamental issues in the interpretation of cosmical constant itself. The possible existence of very small but non-zero cosmological constant revives in these days due to new observation in cosmology.

Due to this fact, there are many phenomenological ansatzes for the cosmological constant more or less justified by physical arguments (see, e.g. [19]). We remark that observational data indicate that  $\lambda \approx 10^{-55} \text{cm}^{-2}$  while particle physics prediction for  $\lambda$  is greater than this value by factor of order  $10^{120}$ . This discrepancy is known as the *cosmological constant problem*. The vacuum energy assigned to  $\lambda$  appears very tiny but not zero. However, there is no really compelling dynamical explanation for the smallness of the vacuum energy at the moment [2] (simple quantum-mechanical calculations yield the vacuum energy much larger [4]).

A positive non-zero cosmological constant helps overcome the age problem, connected on the one side with the hight estimates of the Hubble parameter and with the age of globular clusters on the other side. To explain this apparent discrepancy the point of

view has often been adopted which allows the cosmological constant to vary in time. The idea is that during the evolution of universe the "black" energy linked with cosmological constant decays into the particles causing its decrease.

As it is well-known, the Einstein field equations with a non-zero  $\lambda$  can be rearranged so that their right-hand sides consist of two terms: the stress-energy tensor of the ordinary matter and an additional tensor

$$T_{ij}^{(\nu)} = \left( \frac{c_4 \lambda}{8\pi G} \right) g_{ij} = \Lambda g_{ij}. \quad (2)$$

In common discussions,  $\Lambda$  is identified with vacuum energy because this quantity satisfies the requirements asked from  $\Lambda$ , i.e. (i) it should have the dimension of energy density, and (ii) it should be invariant under Lorentz transformation. The second property is not satisfied for arbitrary systems, e.g. material systems and radiation. Gliner [3] has shown that the energy density of vacuum represents a scalar function of the four-dimensional space-time coordinates so that it satisfies both above requirements. This is why  $\Lambda$  is commonly *identified* with the vacuum energy.

However, there may be generally other quantities satisfying also the above requirements. Instead of identifying  $\Lambda$  with the vacuum energy we have identified  $\Lambda$  in [15] with the stress-energy scalar  $T = T_i^i$  a scalar which arises by the contraction of the stress-energy tensor of the ordinary matter  $T_i^j$ . This quantity *likewise* satisfies both above requirements, i.e., it is Lorentz invariant and has the dimension of the energy density. Hence, we make the ansatz

$$\Lambda_A = \frac{c^4 \lambda_A}{8\pi G} = \kappa T_i^i = \kappa T \quad (3)$$

or

$$\lambda_A = \frac{8\pi G \kappa T}{c^4}, \quad (4)$$

where  $\kappa$  is a dimensionless constant to be determined.  $\Lambda_A$  is a dynamical quantity, often changing over time, representing, in the quintessence theory, the quintessence component. In contrast with some other cosmological models, we suppose that the universe consists of a mixture of the *ordinary* mass-energy and the quintessence component functionally

linked with  $T$  via the cosmological constant  $\lambda_A$ . We note that there are similar attempts to identify  $\lambda$  with the Ricci scalar (see [24]).

In what follows, we introduce a cosmological model with the additional cosmical term  $\Lambda = \kappa T$ . The constant  $\kappa$  is specified by the assumption that the energy density of the universe is equal to its critical value. We show that in the matter-dominated universe the evolution of the scale factor in this model is in matter-dominated universe determined by the density parameter  $\Omega_M$  in a relatively simple way.

## 2 Friedmann's model with a $\Omega_M$ -dependent cosmological constant

The standard Einstein field equations are (see, e.g. [28])

$$R_{ij} - g_{ij}(1/2)R - \lambda g_{ij} = \frac{8\pi G}{c^4} T_{ij}^{(m)}. \quad (5)$$

These equations can be rewritten in the form

$$R_{ij} - g_{ij}(1/2)R = \frac{8\pi G}{c^4} (T_{ij}^{(m)} + T_{ij}^{(v)}), \quad (6)$$

where

$$T_{ij}^{(v)} = g_{ij}\Lambda \quad \Lambda = \frac{\lambda c^4}{8\pi G}.$$

Putting  $\Lambda = \Lambda_A = \kappa T$  we have

$$T_{ij}^{(v)} = g_{ij}\kappa T.$$

and Eq.(6) becomes

$$R_{ij} - g_{ij}(1/2)R = \frac{8\pi G}{c^4} [T_{ij}^{(m)} + g_{ij}\kappa T]. \quad (7)$$

In a homogeneous and isotropic universe characterized by the Friedmann-Robertson-Walker line element the Einstein equations with matter in the form of a perfect fluid and non-zero cosmical term  $\lambda$  acquire the following form

$$\frac{3\dot{R}(t)^2}{R(t)^2} = 8\pi G\rho + \lambda c^2 - 3\frac{kc^2}{R^2(t)}, \quad (8)$$

and

$$\ddot{R}(t) = \frac{4\pi G}{3}(-\rho - 3p/c^2) + \frac{\lambda c^2}{3}R(t), \quad (9)$$

where  $R(t)$  is the time-dependent scale factor.

To determine the exact form of  $\Lambda_A$  which is to be inserted in Eqs.(8) and (9) we have to specify  $\kappa$  and  $T$ .  $T$  can be derived from the tensor  $T_j^i$  and  $\kappa$  in Eq.(3) we determine by assuming that the universe is flat, i.e.,  $\Omega_{tot} = 1$ . This is consistent with the inflationary cosmology which assumes that the universe is spatially flat and that its total energy density is equal to the critical density ( $\Omega_{tot} = 1$ ). This assumption is also conformed by the current measurement of the cosmic microwave background anisotropy [18]. Since  $\Omega_M < 1$  we suppose that the remaining energy required to produce a geometrical flat universe is given by the equation

$$\Omega_M + \Omega_Q = \Omega_M + \kappa\Omega_M = 1.$$

This gives

$$\kappa = \frac{1}{\Omega_M} - 1. \quad (10)$$

By specifying  $\kappa$  and  $T$ , the cosmological constant  $\Lambda_A$  is uniquely determined so we can investigate the cosmological models with  $\Lambda_A$  for the different values of  $\Omega_M$ .

$$\Lambda_A = \left( \frac{1}{\Omega_M} - 1 \right) T. \quad (11)$$

The stress-energy tensor of the cosmic medium  $T_j^i$  in the everywhere local rest frame has only four non-zero components  $T_0^0 = \varrho c^2, T_1^1 = T_2^2 = T_3^3 = -p$  [5]. Therefore,

$$T = \varrho c^2 - 3p/c^2. \quad (12)$$

Setting  $p = w\rho c^2$  we have

$$T = \varrho c^2(1 - 3w). \quad (13)$$

We see that  $\Lambda_A$  is a function of both the mass-energy density and some *stress* components.

A comprehensive analysis of Eqs.(8) and (9) has been carried out in [27] for static cosmological term, and in [16] for a varying cosmological term. A quantitative analysis

of solutions to Eqs.(8) and (9) can be gained by eliminating  $\rho$  in these equations and combining them into a single equation for the evolution of the scale factor in the presence of a  $\lambda$ -term [26]

$$\frac{2\ddot{R}}{R} + (1 + 3w)\left(\frac{\dot{R}^2}{R^2} + \frac{kc^2}{R^2}\right) - (1 + w)\lambda c^2 = 0, \quad (14)$$

### 3 Matter-dominated epoch

In what follows, we consider the matter-dominated and flat universe, i.e. we set  $w = 0$  and  $k = 0$ . In this universe  $T = \rho_M c^2$ . Inserting  $T = \rho_M c^2$  in the equation for  $\Lambda_A$  we have

$$\Lambda_A = \kappa \rho_M c^2 = \left(\frac{1}{\Omega_M} - 1\right) \rho_M c^2 = (\varrho_{crit} - \varrho_M) c^2 = \varrho_{crit}(1 - \Omega_M) c^2, \quad (15)$$

The critical density  $\varrho_{crit}$  we obtain by inserting Eq.(15) into Eq.(8)

$$\varrho_{crit} = \frac{3\dot{R}^2}{R^2}. \quad (16)$$

The insertion of Eq.(16) into Eq.(14) yields immediately the equation for the evolution of  $R(t)$  in the matter dominated epoch

$$\ddot{R}(t) = \left(1 - \frac{3}{2}\Omega_M(t)\right) \frac{(\dot{R}(t))^2}{R(t)}. \quad (17)$$

In our model, this equation describes the time dependence of the scale factor as a function of  $\Omega_M(t)$  and represents so the basic equation for the evolution dynamics of a pressure-free and flat universe.

The exact solution of (17) can be found for an arbitrary time function  $\Omega_M(t)$ . With the ansatz  $R = \exp(y)$  we have

$$\dot{R} = \dot{y} \exp(y), \quad \ddot{R} = (\ddot{y} + (\dot{y})^2) \exp(y)$$

which inserting into Eq.(17) yields

$$-(2/3)\Omega_M(t)(\dot{y})^2 = \ddot{y}.$$

By putting  $\dot{y} = q$ , this equation becomes the form

$$-(2/3)\Omega_M(t) = \frac{\dot{q}}{q^2},$$

the solution to which is

$$q = \frac{1}{\int (2/3)\Omega_M(t)dt + C_1}.$$

Since  $\dot{y} = q$  we have

$$y = \int \left( \frac{1}{\int (2/3)\Omega_M(t)dt + C_1} \right) dt + C_2.$$

With  $y(t)$ , the general solution of Eq.(17) is

$$R(t) = \exp \int \left( \frac{1}{\int (2/3)\Omega_M(t)dt + C_1} \right) dt + C_2. \quad (18)$$

We see that evolution of  $R(t)$  in a pressure-free medium is a function of the time dependence of  $\Omega_M(t)$  and the integration constants  $C_1$  and  $C_2$ .

## 4 Evolution of $R(t)$ for the different density parameters

We present solutions of Eq.(17) for some selected constant values of  $\Omega_M$  and derive then the cosmological parameters of the corresponding models of universe.

(A) For  $\Omega_M = 0$ , i.e. for a massless universe, we get a typical inflationary solution of Eq.(17)

$$R(t) = \exp(C_1(t - C_2)).$$

(B) For  $\Omega_M = 2/3$ , the evolution of  $R(t)$  is

$$R(t) = C_1 + tC_2.$$

With  $C_1 = 0$ , it gives

$$R(t) = C_2t.$$

(C) For  $\Omega_M = 1/3$ , the evolution of  $R(t)$  is

$$R(t) = \frac{1}{4}C_1(t^2 - 2tC_2 + C_2^2)^{1/3}.$$

With  $C_2 = 0$ , it gives

$$R(t) = \frac{1}{4}C_1t^2.$$

(D) For  $\Omega_M = 1$ , the evolution of  $R(t)$  is

$$R(t) = \frac{C_1^{2/3} (9t^2 - 18t(C_2 + 9C_2^2))^{1/3}}{2^{2/3}}.$$

With  $C_2 = 0$ , we obtain  $R(t) = Kt^{2/3}$ , i.e. the evolution law which is identical with that of the Standard Cosmology in a pressure-free cosmic medium. We see that in all cases (except A)  $R(t)$  satisfies the initial condition  $R(0) = 0$  and represents a smoothly increasing functions of time.

We now analyse the cases A, B, C, D in more details.

**Case A.** It is tempting to choose for the early universe  $\Omega_M = 0$ , i.e. to suppose that the universe started in a massless state and its mass content was created later through the decay of the cosmical term. Under this assumption we have

$$R(t) = \exp(C_1(t - C_2)) = R_0 \exp(C_1 t), \quad C_1 = \frac{1}{t_0}. \quad (19)$$

The natural measures for length and time in cosmology is the Planck length and time, i.e.,  $l_p = (Gh/c^3)^{1/2} = 4.3 \cdot 10^{-35} \text{ m}$  and  $t_p = (Gh/c^5)^{1/2} = 1.34 \cdot 10^{-43} \text{ s}$ , respectively. It is reasonable to assume that at the very beginning of the cosmic evolution the radius of the universe was of the order of the Planck length, therefore we put in Eq.(19) the integration constant  $R_0$  and  $C_1$  equal to  $l_p$  and  $1/t_p$ , respectively. Then, we get for the initial radius and the velocity the values

$$R(0) = l_p = 4.3 \cdot 10^{-35} \text{ m}. \quad \dot{R}(0) = \frac{l_p}{t_p} = c = 3 \cdot 10^8 \text{ m s}^{-1},$$

respectively. The most interesting thing of the A-type universe is its inflationary character (for a recent review see [14]).

**Case B.** In the B-type universe the relevant cosmic parameters are

$$R = C_2 t, \quad \dot{R}(t) = C_2, \quad H = \frac{1}{t} \quad q = 0.$$

The age of this universe, if taking  $H = 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ , is

$$t_0 = \frac{1}{H_0} = 2 \cdot 10^{10} \text{ yr}.$$

This age is larger than that in the Standard Model. However, this universe is not accelerating so it seems not to be compatible with the recent data.

**Case C.** The relevant cosmological parameters here are

$$R(t) = (1/4)C_1 t^2 \quad \dot{R} = \frac{C_1 t}{2}, \quad q = -\frac{1}{2}, \quad H(t) = \frac{2}{t} \quad \lambda_A = \frac{8}{c^2 t^2}.$$

In the C-type universe there is no age problem because  $t_0 = 2/H_0$ . The age of this universe is approximately  $4.10^{10} \text{ yr}$ , i.e. old enough for the evolution of the globular clusters. This universe is accelerated ( $q = -0.5$ ). Its density parameter is  $\Omega_M = 1/3$  which corresponds to the recent data. In this universe the proper distance  $L(t)$  to the horizon, which is the linear extent of the causally connected domain, diverges

$$L(t) = R(t) \int_0^t \frac{d\tau}{R(\tau)} = C_2 t^2 \left[ -\left(\frac{2}{\tau}\right) \right]_0^t = -\infty,$$

In [29] is shown that the only way to make the whole of the observable universe causally connected is to have a model with infinite  $L(t)$  for all  $t > 0$ , i.e. the whole C-type observable universe is causally connected. It is noteworthy that the decay law for the cosmical constant of the form  $\lambda = at^{-2}$ , was phenomenologically set by several authors whereby different authors used different physical arguments for its justification [18]- [25]. (For a recent review see [16]).

**Case D.** In this universe we have

$$R(t) = K t^{2/3} = \frac{C_1^{2/3} 9}{2^{2/3}} t^{2/3}.$$

All other parameters of the D-type universe are identical with those of Standard Model.

In order to vanish the covariant divergence of the right-hand side of Eq.(6) the matter is created along with energy and momentum. Therefore, the cosmological constant  $\lambda_A$  decays during the cosmological time and new particles are created. The present rate of matter creation in the matter dominated epoch is very small [29]

$$n = \frac{1}{R^3} \frac{d(\rho R^3)}{dt} \Big|_0.$$

## 5 Final remark

Summing up we can state:

- (i) In previous sections we have shown that the density parameter  $\Omega_M$  determines, in our model of the universe, its entire evolution dynamics. In the basic dynamical equation (17) the energy density does not explicitly appear only in the density parameter  $\Omega_M$ . We note that the density parameter  $\Omega_M$  as the ratio of  $\rho_M$  and  $\rho_{crit}$  may be *finite* although both quantities are infinite.
- (ii) There is growing observational evidence that the total matter of the universe is significantly less than the critical density. Several authors [7] [8] [9] have found that the best and simplest fit is provide by ( $h = 0.65 \pm 0.15$ )

$$\Omega_M = \Omega_{CDM} + \Omega_{baryon} \approx [0.30 \pm 0.10] + [0.04 \pm 0.01] \approx 1/3$$

which is approximately the density parameter considered in the C-type universe. [11].

- (iii) In the recently popular  $\Lambda$ CDM cosmological model, which consists of a mixture of vacuum energy and cold dark matter, a serious problem exists called in [12] as the cosmic coincidence problem. Since the vacuum energy density is constant over time and the matter density decreases as the universe expands it appears that their ratio must be set to immense small value ( $\approx 10^{-120}$ ) in the early universe in order for the two densities to nearly coincide today, some billions years later. No coincidence problem exists in the C-type universe because  $\Lambda_A$  here is functionally connected with  $\Omega_M$  in such a way that this ratio in the matter dominated epoch does not vary over time.

- (iv) In the radiation dominated epoch  $w = 1/3$  and, according to Eq.(13,  $T = 0$ ). The evolution dynamics in this epoch runs so as if  $\lambda = 0$ .

In conclusion, when comparing the cosmological parameters of the different cosmological models we see that the recent observational data of the flat and acceleration universe are most consistent with the C-type universe. This universe is light-weight, is strictly flat, is accelerating, is old enough and is causally connected.

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